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Self-dual Yang–Mills fields in pseudo-Euclidean spaces

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Abstract
The self-duality Yang–Mills equations in pseudo-Euclidean spaces of dimensions $d \leqslant 8$ are investigated. New classes of solutions of the equations are found. Extended solutions to the $D = 10$, $N = 1$ supergravity and super Yang–Mills equations are constructed from these solutions.

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1. Introduction

In 1983 Corrigan *et al* [1] proposed a generalization of the self-dual Yang–Mills equations in dimension $d > 4$:

$$f_{mnps} F^{ps} = \lambda F_{mn} \tag{1}$$

where the numerical tensor f_{mnps} is completely antisymmetric and $\lambda = \text{const}$ is a nonzero eigenvalue. By the Bianchi identity $D_{[p} F_{mn]} = 0$, it follows that any solution of (1) is a solution of the Yang–Mills equations $[D^m, F_{mn}] = 0$. Some of these solutions can be found in [2].

The many-dimensional Yang–Mills equations appear in the low-energy effective theory of superstrings and supermembranes [3, 4]. In addition, there is a hope that Higgs fields and supersymmetry can be understood through dimensional reduction from $d > 4$ dimensions down to $d = 4$ [5].

The paper is organized as follows. Section 2 contains well-known facts about Cayley–Dickson algebras and Lie algebras connected with them. In sections 3 and 4 the self-duality Yang–Mills equations in pseudo-Euclidean spaces of dimensions $d \leqslant 8$ are investigated. In section 5 extended solutions to the $D = 10$, $N = 1$ supergravity and super Yang–Mills equations are constructed from these solutions.

2. Cayley–Dickson algebras

Let us recall that the algebra A satisfying the identities

$$x^2y = x(xy) \qquad yx^2 = (yx)x \tag{2}$$

is called alternative. It is obvious that any associative algebra is alternative. The most important example of nonassociative alternative algebra is the Cayley–Dickson algebra. Let us recall its construction (see [6]).

Let A be an algebra with an involution $x \to \bar{x}$ over a field F of characteristic $\neq 2$. Given a nonzero $\alpha \in F$ we define a multiplication on the vector space $(A, \alpha) = A \oplus A$ by

$$(x_1, y_1)(x_2, y_2) = (x_1x_2 - \alpha\bar{y}_2y_1, y_2x_1 + y_1\bar{x}_2).$$

This makes (A, α) an algebra over F. It is clear that A is isomorphically embedded in (A, α) and $\dim(A, \alpha) = 2\dim A$. Let $e = (0, 1)$. Then $e^2 = -\alpha$ and $(A, \alpha) = A \oplus Ae$. Given any $z = x + ye$ in (A, α) we suppose $\bar{z} = \bar{x} - ye$. Then the mapping $z \to \bar{z}$ is an involution in (A, α).

Starting with the base field F the Cayley–Dickson construction leads to the following sequence of alternative algebras:

(1) F, the base field.
(2) $\mathbb{C}(\alpha) = (F, \alpha)$, a field if $x^2 + \alpha$ is the irreducible polynomial over F; otherwise, $\mathbb{C}(\alpha) \simeq F \oplus F$.
(3) $\mathbb{H}(\alpha, \beta) = (\mathbb{C}(\alpha), \beta)$, a generalized quaternion algebra. This algebra is associative but not commutative.
(4) $\mathbb{O}(\alpha, \beta, \gamma) = (\mathbb{H}(\alpha, \beta), \gamma)$, a Cayley–Dickson algebra. Since this algebra is not associative, the Cayley–Dickson construction ends here.

The algebras in (1)–(4) are called composition. Any of them has the non-degenerate quadratic form (norm) $n(x) = x\bar{x}$, such that $n(xy) = n(x)n(y)$. In particular, over the field $\mathbb{R}$ of real numbers, the above construction gives three split algebras (e.g. if $\alpha = \beta = \gamma = -1$) and four division algebras (if $\alpha = \beta = \gamma = 1$): the fields of real $\mathbb{R}$ and complex $\mathbb{C}$ numbers, the algebras of quaternions $\mathbb{H}$ and octonions $\mathbb{O}$, taken with the Euclidean norm $n(x)$. Note also that any simple nonassociative alternative algebra is isomorphic to the Cayley–Dickson algebra $\mathbb{O}(\alpha, \beta, \gamma)$.

Let A be the Cayley–Dickson algebra and $x \in A$. Denote by R_x and L_x the operators of right and left multiplication in A

$$R_x : a \to ax \qquad L_x : a \to xa.$$

It follows from (2) that

$$R_{ab} - R_aR_b = [R_a, L_b] = [L_a, R_b] = L_{ba} - L_aL_b. \tag{3}$$

Consider the Lie algebra $\mathcal{L}(A)$ generated by all operators R_x and L_x in A. Choose in $\mathcal{L}(A)$ the subspaces $R(A)$, $S(A)$ and $D(A)$ generated by the operators R_x, $S_x = R_x + 2L_x$ and $2D_{x,y} = [S_x, S_y] + S_{[x,y]}$ respectively. Using (3), it is easy to prove that

$$3[R_x, R_y] = D_{x,y} + S_{[x,y]} \tag{4}$$

$$[R_x, S_y] = R_{[x,y]} \tag{5}$$

$$[R_x, D_{y,z}] = R_{[x,y,z]} \tag{6}$$

$$[S_x, S_y] = D_{x,y} - S_{[x,y]} \tag{7}$$

$$[S_x, D_{y,z}] = S_{[x,y,z]} \tag{8}$$

$$[D_{x,y}, D_{z,t}] = D_{[x,z,t],y} + D_{x,[y,z,t]} \tag{9}$$

where $[x, y, z] = [x, [y, z]] - [y, [z, x]] - [z, [x, y]]$. It follows from (4)–(9) that the algebra $\mathcal{L}(A)$ is decomposed into the direct sum

$$\mathcal{L}(A) = R(A) \oplus S(A) \oplus D(A)$$

of the Lie subalgebras $D(A)$, $D(A) \oplus S(A)$ and the vector space $R(A)$.

In particular, if A is a real division algebra, then $D(A)$ and $D(A) \oplus S(A)$ are isomorphic to the compact Lie algebras g_2 and $so(7)$ respectively. If A is a real split algebra, then $D(A)$ and $D(A) \oplus S(A)$ are isomorphic to noncompact Lie algebras g_2' and $so(4, 3)$.

3. Self-dual solutions in $d = 8$

Let A be a real linear space equipped with a nondenerate symmetric metric g of signature $(8, 0)$ or $(4, 4)$. Choose the basis $\{1, e_1, \ldots, e_7\}$ in A such that

$$g = \mathrm{diag}(1, -\alpha, -\beta, \alpha\beta, -\gamma, \alpha\gamma, \beta\gamma, -\alpha\beta\gamma) \tag{10}$$

where $\alpha, \beta, \gamma = \pm 1$. Define the multiplication

$$e_i e_j = -g_{ij} + c_{ij}{}^k e_k \tag{11}$$

where the structural constants $c_{ijk} = g_{ks} c_{ij}{}^s$ are completely antisymmetric and different from 0 only if

$$c_{123} = c_{145} = c_{167} = c_{246} = c_{275} = c_{374} = c_{365} = 1. \tag{12}$$

The multiplication (11) transforms A into a linear algebra. It can easily be checked that the algebra $A \simeq \mathbb{O}(\alpha, \beta, \gamma)$. In the basic $\{1, e_1, \ldots, e_7\}$ the operators

$$R_{e_i} = e_{i0} + \tfrac{1}{2} c_i{}^{jk} e_{jk} \qquad L_{e_i} = e_{i0} - \tfrac{1}{2} c_i{}^{jk} e_{jk} \tag{13}$$

where e_{ij} are generators of the Lie algebra $\mathcal{L}(A)$ satisfying the switching relations

$$[e_{mn}, e_{ps}] = g_{mp} e_{ns} - g_{ms} e_{np} - g_{np} e_{ms} + g_{ns} e_{mp}. \tag{14}$$

Now, let G be a matrix Lie group constructed by the Lie algebra $D(A) \oplus S(A)$. In the space A equipped with the metric (10) we define the completely antisymmetric G-invariant tensor f_{mnps} (cf [7]):

$$f_{ijk0} = c_{ijk} \qquad f_{ijkl} = g_{il} g_{jk} - g_{ik} g_{jl} + c_{ijm} c_{kl}{}^m$$

where $i, j, k, l \neq 0$. Representing the nonzero components of f_{mnps} in the explicit form

$$f_{0123} = f_{0145} = f_{0167} = f_{0246} = f_{0275} = f_{0374} = f_{0365} = 1$$

$$f_{4567} = f_{2367} = f_{2345} = f_{1357} = f_{1364} = f_{1265} = f_{1274} = 1$$

we see that the tensor f_{mnps} satisfies the identity

$$f_{mnij} f_{ps}{}^{ij} = 6(g_{mp} g_{ns} - g_{ms} g_{np}) + 4 f_{mnps}. \tag{15}$$

Define the projectors $\tilde{f}_{mnps}$ of $\mathcal{L}(A)$ onto the subalgebra $D(A) \oplus S(A)$ and its generators $\tilde{f}_{mn}$ by

$$\tilde{f}_{mnps} = \tfrac{3}{8}(g_{mp} g_{ns} - g_{ms} g_{np}) - \tfrac{1}{8} f_{mnps} \qquad \tilde{f}_{mn} = \tilde{f}_{mn}{}^{ij} e_{ij}.$$

It follows from (15) that

$$f_{mnij} \tilde{f}_{ps}{}^{ij} = -2 \tilde{f}_{mnps} \tag{16}$$

$$f_{mnij} \tilde{f}^{ij} = -2 \tilde{f}_{mn}. \tag{17}$$

Using the identities (7)–(9) and (13), we get the switching relations

$$[\tilde{f}_{mn}, \tilde{f}_{ps}] = \tfrac{3}{4}(\tilde{f}_{m[p}g_{s]n} - \tilde{f}_{n[p}g_{s]m}) - \tfrac{1}{8}\big(f_{mn}{}^{k}{}_{[p}\tilde{f}_{s]k} - f_{ps}{}^{k}{}_{[m}\tilde{f}_{n]k}\big). \tag{18}$$

Now we can find solutions of (1). We choose the ansatz (cf [2])

$$A_m(x) = \frac{4}{3}\frac{\tilde{f}_{mi}x^i}{\rho^2 + r^2} \tag{19}$$

where $r^2 = x_k x^k$ and $\rho \in \mathbb{R}$. Using the switching relations (18), we get

$$F_{mn}(x) = -\frac{4}{9}\frac{(6\rho^2 + 3r^2)\tilde{f}_{mn} + 8\tilde{f}_{mni}{}^{s}\tilde{f}_{sj}x^i x^j}{(\rho^2 + r^2)^2}. \tag{20}$$

It follows from (16), (17) that the tensor F_{mn} is self-dual. If the metric (10) is Euclidean, then we have the well-known solution of equations (1) (see [2]). If the metric (10) is pseudo-Euclidean, then we have a new solution.

4. Solutions in $d < 8$

Now we will find solutions of the self-duality equations in dimension $d < 8$. If B_α is a subalgebra of the real Cayley–Dickson algebra A, then the subgroup H_α of automorphisms of A leaving the element of B_α fixed is called the Galois group of A over B_α. A description of these groups is known [8]. In particular, if A is the real division algebra and $B_1 \simeq \mathbb{R}$, $B_2 \simeq \mathbb{C}$, $B_3 \simeq \mathbb{H}$, then

$$G \simeq \mathrm{Spin}(7) \qquad H_1 \simeq G_2 \qquad H_2 \simeq SU(3) \qquad H_3 \simeq SU(2).$$

If A is the real split algebra, then for the same choice of B_i,

$$G \simeq \mathrm{Spin}(4,3) \qquad H_1 \simeq G_2' \qquad H_2 \simeq SU(2,1) \qquad H_3 \simeq SU(1,1).$$

Obviously, the orthogonal complement $B_\alpha^\perp$ of B_α in A is the H_α-invariant subspace of dimension $d_\alpha = 8 - \dim H_\alpha$. Now it is easy to construct a completely antisymmetric H_α-invariant d_α-tensor f^α_{mnps}. It is a projection of the d-tensor $f_{mnps} \in \Lambda^4(A)$ onto the subspace $\Lambda^4(B_\alpha)$. We can choose nonzero components of f_{mnps} in the form

$$f^1_{4567} = f^1_{2367} = f^1_{1274} = f^1_{1357} = f^1_{1364} = f^1_{1265} = f^1_{2345} = 1$$
$$f^2_{1364} = f^2_{1265} = f^2_{2345} = 1$$
$$f^3_{2345} = 1.$$

Now we can easily get analogues of the identities (15)–(18). In particular, the switching relations (18) take the form

$$\big[\tilde{f}^\alpha_{mn}, \tilde{f}^\alpha_{ps}\big] = \frac{3-\alpha}{4-\alpha}\big(\tilde{f}^\alpha_{m[p}g_{s]n} - \tilde{f}^\alpha_{n[p}g_{s]m}\big) - \frac{1}{8-2\alpha}\big(f^{\alpha}_{mn}{}^{k}{}_{[p}\tilde{f}^\alpha_{s]k} - f^{\alpha}_{ps}{}^{k}{}_{[m}\tilde{f}^\alpha_{n]k}\big).$$

Note that if we choose the ansatz $A_m(x)$ in the form

$$A_m(x) = k\frac{\tilde{f}^\alpha_{mi}x^i}{\rho^2 + r^2}$$

then the corresponding field strength F_{mn} is not self-dual for $\alpha = 2$. In contrast, if $\alpha = 1$ or $\alpha = 3$, then the choice of $A_m(x)$ in the form (21) reduces to a self-dual field strength. For example, if $\alpha = 1$, then $k = 3/2$ and

$$F_{mn}(x) = -\frac{3}{2}\frac{(2\rho^2 + r^2)\tilde{f}^1_{mn} + 3\tilde{f}^1_{mni}{}^{s}\tilde{f}^1_{sj}x^i x^j}{(\rho^2 + r^2)^2}. \tag{21}$$

For a Euclidean metric this solution is known (see [2]). For a pseudo-Euclidean metric we have a new solution. For $\alpha = 3$ we have the well-known instanton solution [9] or its noncompact analogue.

Note that in $d = 4$ there exists another solution of the Yang–Mills equations. It depends on coordinates of the Minkowski space. Indeed, we choose the ansatz $A_m(x)$ in the form

$$A_m(x) = \frac{2e_{mn}x^n}{\lambda^2 + x_k x^k} \tag{22}$$

where e_{mn} are generators of the Lie algebra $so(p, q)$ satisfying the relations $so(p, q)$. Then the field strength

$$F_{mn}(x) = \frac{-4\lambda^2\, \mathrm{e}_{mn}}{(\lambda^2 + x_k x^k)^2} \tag{23}$$

and

$$\partial^m F_{mn} + [A^m, F_{mn}] = \frac{8\lambda^2\, \mathrm{e}_{mn}x^m}{(\lambda^2 + x_k x^k)^3}\left(4 - \delta^i_i\right).$$

Hence the ansatz (22) satisfies the Yang–Mills equations if $p + q = 4$. If $|p - q| = 4$ or 0, then the algebra $so(p, q)$ is isomorphic to the direct sum $so(3) \oplus so(3)$ or $so(2, 1) \oplus so(2, 1)$ of proper subalgebras. Therefore projecting $A_m(x)$ on these subalgebras, we again get the instanton solution [9] or its noncompact analogue. If $|p - q| = 2$, then the algebra $so(p, q)$ is simple. In this case the solution (23) of the Yang–Mills equations is not self-dual.

5. Extended supersymmetric solutions

Let us now show that the above instanton solutions can be extended to solutions of the $D = 10$, $N = 1$ supergravity and super Yang–Mills equations. Consider the purely bosonic sector of the theory

$$S = \frac{1}{2k^2}\int \mathrm{d}^{10}x\sqrt{-g}\,\mathrm{e}^{2\phi}\left(R + 4(\nabla)^2 - \frac{1}{3}H^2 - \frac{\alpha'}{30}\mathrm{Tr}(F^2)\right). \tag{24}$$

Rather than directly solving the equations of motion for this action, it is much more convenient to look for bosonic backgrounds which are annihilated by some of $N = 1$ supersymmetry transformations. This requires that in ten dimensions there exists at least one Majorana–Weyl spinor ϵ such that the super symmetry variations

$$\delta\chi = F_{MN}\Gamma^{MN}\epsilon$$
$$\delta\lambda = \left(\Gamma^M\partial_M\phi - \tfrac{1}{6}H_{MNP}\Gamma^{MNP}\right)\epsilon$$
$$\delta\psi_M = \left(\partial_M + \tfrac{1}{4}\left(\omega_M^{AB} - H_M^{AB}\right)\Gamma_{AB}\right)\epsilon$$

of the fermionic fields vanish for such solutions. We will construct a simple ansatz for the bosonic fields which does just this (cf [3]).

First, we choose ϵ to be the Spin(4, 3)-singlet of the Majorana–Weyl spinor of $SO(5, 5)$. Denote world indices of the eight-dimensional subspace indices by $\mu, \nu = 1, \ldots, 8$ and the corresponding tangent space indices by $m, n = 1, \ldots, 8$. We assume that no fields depend on the coordinates with indices $M, N = 0, 9$. It follows that

$$\tilde{f}_{mnps}\Gamma^{ps}\epsilon = \tilde{f}_{mn}\epsilon = 0.$$

Using expression (20) for the tensor field strength F_{mn}, we see that the supersymmetry variation $\delta\chi$ vanishes.

Further, we choose the antisymmetric tensor field strength H_{mnp} and metric tensor g in the form

$$H_{mnp} = \frac{1}{7} f_{mnps} \partial^s \phi \qquad g_{\mu\nu} = e^{(6/7)\phi} g_{0\mu\nu} \tag{25}$$

where g_0 is the pseudo-Euclidean metric (10). Using the identities

$$f_{mnps} \Gamma^{mnp} = 42 \Gamma_s$$

and the explicit form of spin connectedness

$$\omega_{\mu mn} = \frac{3}{7} (\delta_{\mu m} \partial_n \phi - \delta_{\mu n} \partial_m \phi)$$

we prove that the variations $\delta\lambda$ and $\delta\psi_M$ also vanish for any $\phi(x)$.

It follows from the Bianchi identity

$$\partial_{[m} H_{nps]} = -\alpha' \mathrm{Tr}_8 F_{[mn} F_{ps]}$$

that the tensor field strength

$$H_{mnp} = -\alpha' \frac{3\rho^2 + r^2}{9(\rho^2 + r^2)^3} f_{mnps} x^s. \tag{26}$$

If we compare (26) with (25), we find

$$\mathrm{e}^{(6/7)\phi} = e^{(6/7)\phi_0} + \alpha' \frac{2\rho^2 + r^2}{3(\rho^2 + r^2)^2} \tag{27}$$

where ϕ_0 is the value of the dilaton ϕ on ∞. Similarly, if we choose the G_2'-singlet of the Majorana–Weyl spinor of $SO(5,5)$ and use expression (21) for the tensor field strength F_{mn}, we get an analogue of the solution (27).

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